Theoretical analysis for the reflection of a focused ultrasonic beam from a fluid–solid interface

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A theoretical study of the reflection of focused acoustic beams from a fluid–solid interface is presented. The incident field is defined by a Gaussian velocity distribution along a plane emitter. The reflected field is described through its pressure field in a region not restricted to the interface. Nonspecular phenomena were exhibited at any angle of incidence by means of short wave asymptotic analysis. In particular, for incidence near the Rayleigh angle, a distortion of a part of the caustic of the reflected beam including lateral and axial displacements of the focal point is observed.

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INTRODUCTION

Experimental studies[1] on the reflection of a bounded acoustic beam incident on a plane water–metal interface have shown that, for incidence at or near the Rayleigh angle, the reflected profile exhibits an unexpectedly large width, a silent or minimum intensity zone, and a lateral shift of the maximum intensity. The general features of the phenomenon have been described by Bertoni and Tamir[2] in 1973, as the result of the superposition of two parts: the usual geometric reflected beam and the acoustic field generated by reradiation of the leaky Rayleigh wave. Numerical calculation[3] of the reflected beam profiles yield similar results and an extension of the theory has been proposed by Rousseau and Gatignol.[4]

The interest in use of ultrasonic focused beams for NDE applications, particularly in the case of the reflection acoustic microscope, has led to many recent studies on the reflection of focused beams. In 1985, Bertoni *et al.*[5] studied the reflection of convergent beams on a liquid–solid interface at the Rayleigh angle incidence using the hypothesis of a well-collimated beam. They obtained the position of the focal point of the reflected beam and predicted both lateral and axial displacements, using an approximation for the reflected acoustic field. In 1986, Nagy *et al.*[6] verified the axial displacement by means of schlieren photography. The developed model[5] has a number of advantages (simplicity and amenability to analytical solution), but is difficult to apply in its present form to beams having more pronounced convergence, and at angles of incidence other than the Rayleigh angle. Further, in the model there is no notion of the caustic of the incident or reflected beams; the only information about the modified reflected field is the displacement of the focal point.

In this paper, we extend the previous theories[2,5] to include the following.

(1) The incident beam is now defined by its normal velocity distribution along a plane emitter placed in the fluid and introduce the notion of the caustic of the acoustic beam.

(2) Initially, an asymptotic evaluation of the reflected pressure field was obtained for any angle of incidence, in particular for the Rayleigh angle, using short wave assumptions by means of a steepest descent procedure.[7,8] We observed that for incidence near the Rayleigh angle where the phase of the reflection coefficient varies abruptly, contrary to the case of bounded beams,[4] this asymptotic method is not applicable. The physical significance of this observation is the following: the presence of the Rayleigh pole of the reflection coefficient (singularity in the complex plane relative to a generalized Rayleigh wave), does not affect the reflected beam; i.e., the reradiation of a leaky Rayleigh wave in the fluid medium (which is the cause of nonspecular reflection of a parallel beam) is very low.

(3) Finally, an asymptotic evaluation of the reflected field was obtained by means of the stationary phase method[7,8] applied to the Fourier representations. This allows one to explore the reflected pressure in a region that is not limited to the interface alone, and thus obtain a spatial representation of the reflected beam.

(4) A distortion of the caustic of the reflected beam was observed in the neighborhood of the Rayleigh angle of incidence, including lateral and axial displacements of the focal point of the beam. At the Rayleigh incidence, the lateral displacement is maximum. At this particular angle, the axial displacement is negligible in comparison to the length of the focal spot and thus experimentally not detectable. Moreover, other nonspecular phenomena are predicted for incidence near the Rayleigh angle: spreading of the reflected beam, asymmetric variation of the acoustic pressure around the axis, and curvature of the acoustic axis.

(5) The nonspecular reflection of a focused beam due to

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the generation of the Rayleigh wave occurs for any angle of incidence. For incidence near the Rayleigh angle, the whole acoustic axis and a part of the caustic are distorted, including axial and lateral displacements of the focal point; for another incidence, a different part of the reflected beam would be modified.

I. THE GAUSSIAN FOCUSED MODEL

A. Definition of the problem

In order to describe the nonspecular reflection of focused acoustic beams incident on plane fluid–solid interfaces, we assume that the incident field is given by a Gaussian distribution of the normal velocity along the plane of the emitter and that the characteristic width of the beam is large compared to the emission wavelength (short wave hypothesis). Thus, the pressure field can be described by means of a Fourier representation; the asymptotic evaluation of the Fourier integral is obtained about the Rayleigh singularity.

Let us consider the configuration of Fig. 1. The half-space $z<0$ is filled with a fluid with mass density ρ and sound velocity c. The half-space $z>0$ is the elastic solid region with mass density ρ_s and with c_{SL} and c_{ST} as longitudinal and shear wave velocities, respectively. The emitting plane is defined by $z_i=0$ in the fluid region. The Gaussian normal velocity distribution along the emitter plane is given by

$$V_n(X_i,0)=V_0 e^{-(x_i/a)^2} e^{-ik\sin\theta_0(x_i^2/a)} e^{-i\omega t}, \qquad (1)$$

where V_0 is the central magnitude of V_n, "a" is the characteristic width of the Gaussian beam, k is the wave number in the fluid given by $k=\omega/c$, with ω the angular frequency of the emitter, and θ_0 is the half-angle of convergent beam.

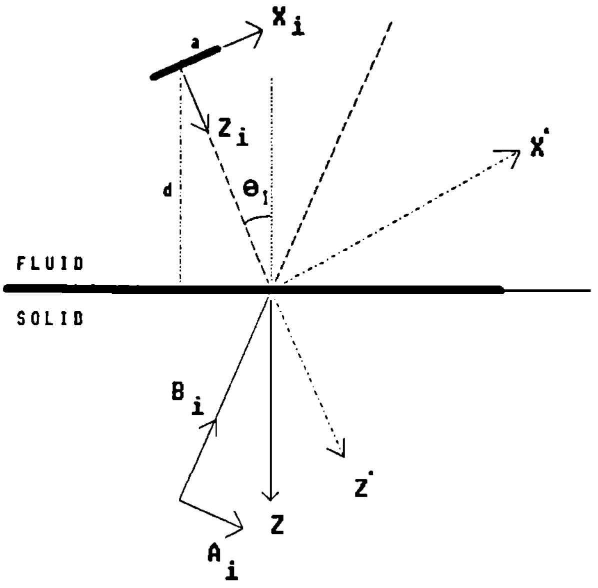


FIG. 1. Configuration of the problem and coordinates definition. (x_i,z_i) is the coordinate system linked to the emitter; (x,z) is the coordinate system linked to the interface, where the continuity conditions are written; (A_i,B_i) is the coordinate system linked to the reflected beam; (x',z') is the coordinate system where the reflected field is expressed (calculus system); and θ_i is the angle of incidence.

B. Incident field

The pressure of the incident field in the coordinate system (x_i,z_i) is given by plane-wave superposition in the form of a Fourier integral:

$$P_{\rm inc}(\bar{x}_i,\bar{z}_i)=\frac{\rho c V_0\sqrt{ka}}{2\sqrt{i\pi}\sin\theta_0}\int_{-\infty}^{+\infty}\frac{\exp[(\bar{k}_{x_i}^2/4\sin^2\theta_0)]}{\bar{k}_{z_i}} \times e^{i(ka)f_i(\bar{k}_{x_i})}\,d\bar{k}_{x_i} \qquad (2)$$

with

$$f_i(\bar{k}_{x_i})=\frac{\bar{k}_{x_i}^2}{4\sin\theta_0}+\bar{x}_i\bar{k}_{x_i}+\bar{z}_i\bar{k}_{z_i}, \qquad (3)$$

where (nondimensional parameters)

$$\bar{x}_i=\frac{x_i}{a},\quad \bar{z}_i=\frac{z_i}{a},\quad \bar{k}_{x_i}=\frac{k_{x_i}}{k},\quad \bar{k}_{z_i}=\frac{k_{z_i}}{k},$$

k_{x_i} and k_{z_i} are the wave-number components along the axes x_i and z_i, respectively.

The function $\bar{k}_{x_i}=\sqrt{1-\bar{k}_{z_i}^2}$ is chosen as real positive when $|\bar{k}x_i|<1$ and imaginary positive when $|\bar{k}x_i|>1$ (according to propagation conditions); this function has two branch points at $\bar{k}x_i=\pm 1$. The real integration path in the system (x_i,z_i), is indicated in Fig. 2.

Using the asymptotic method of steepest descent, the pressure field (2) can be evaluated as (see Refs. 9 and 10 for detailed equations):

$$P_{\rm inc}(\bar{x}_i,\bar{z}_i) = \text{const} * \sum_n(\text{term depending on } \gamma_n), \quad n=1 \text{ or } 3, \qquad (4)$$

where γ_n are the saddle points of the function f_i given by the equation $f_i'(\gamma_n)=0$. Equation (4) can be interpreted in terms of rays; the propagation around a point $(\bar{x}_{i0},\bar{z}_{i0})$ can be locally assimilated to a plane wave of wave number:

$$[k\gamma_n,k\sqrt{1-\gamma_n^2}],\quad n=1 \text{ or } 3.$$

In these conditions, the direction of propagation is given by the straight line defined by the equation:

$$D(\gamma_n):\quad (\bar{x}_i-\bar{x}_{i0})\sqrt{1-\gamma_n^2}-(\bar{z}_i-\bar{z}_{i0})\sqrt{1-\gamma_n^2}=0. \qquad (5)$$

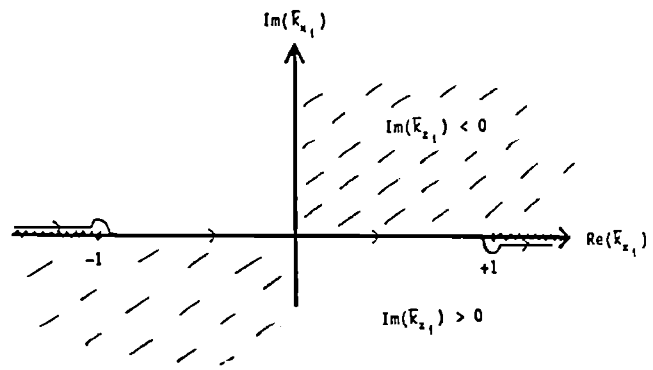


FIG. 2. Real integration path for the incident pressure integral representation, in the system (x_i,z_i).

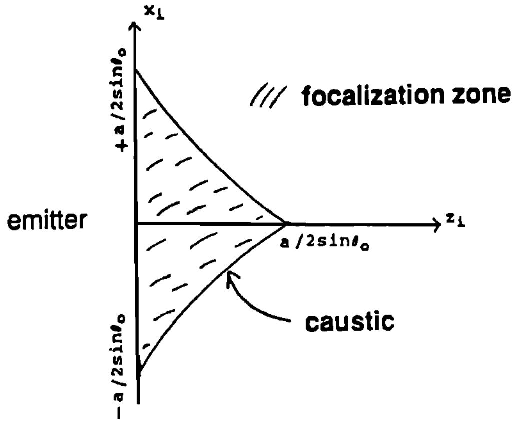


FIG. 3. Incident beam. Focalization zone.

The envelop of the set of all such lines defines the caustic of the field, which can be found from the equations $f_i' = f_i'' = 0$. The caustic devises the physical space in two regions (see Fig. 3).

(i) In the region outside the focal zone, only a single ray passes through each point (case of a single stationary point); Eq. (4) has only one term.

(ii) Within the focal zone, three rays pass through each point; two of them are tangent to the "near" branch of the caustic, the third, to the "further" branch (case of three stationary points); Eq. (4) has three terms.

C. Reflected field

The incident field expressed in the (x,z) coordinate system is given by

$$P_{\text{inc}}(\bar{x},\bar{z}) = \frac{\rho c V_0 \sqrt{ka}}{2\sqrt{i\pi}\sin\theta_0} \int_{-\infty}^{+\infty} \frac{\exp\{-[(\bar{k}_x \cos\theta_I - \bar{k}_z \sin\theta_I)^2/4\sin^2\theta_0]\}}{\bar{k}_z} e^{i(ka)\bar{f}_i(\bar{k}_x)} \, d\bar{k}_x \tag{6}$$

with

$$\bar{f}_i(\bar{k}_x) = \frac{(\bar{k}_x \cos\theta_I - \bar{k}_z \sin\theta_I)^2}{4\sin\theta_0} + (\bar{x} + \bar{d}\tan\theta_I)\bar{k}_x + (\bar{d} + \bar{z})\bar{k}_z, \tag{7}$$

where "d" is the distance between the emitter and the interface, and the bar indicates nondimensional parameters.

Since the incident field is composed of plane waves, writing the continuity conditions in the (x,z) coordinate system is the same as multiplying the integrand of Eq. (6) by the plane-wave reflection coefficient for a fluid–solid interface.

Thus, the pressure of the reflected field is given by

$$P_{\text{ref}}(\bar{x},\bar{z}) = \frac{\rho c V_0 \sqrt{ka}}{2\sqrt{i\pi}\sin\theta_0} \int_{-\infty}^{+\infty} R(\bar{k}_x) \times \frac{\exp\{-[(\bar{k}_x \cos\theta_I - \bar{k}_z \sin\theta_I)^2/4\sin^2\theta_0]\}}{\bar{k}_z} \times e^{i(ka)\bar{f}_r(\bar{k}_x)} \, d\bar{k}_x \tag{8}$$

with

$$\bar{f}_r(\bar{k}_x) = \frac{(\bar{k}_x \cos\theta_I - \bar{k}_z \sin\theta_I)^2}{4\sin\theta_0} + (\bar{x} + \bar{d}\tan\theta_I)\bar{k}_x + (\bar{d} - \bar{z})\bar{k}_z. \tag{9}$$

Here, $R(\bar{k}_x)$ is the plane-wave reflection coefficient:

$$R(\bar{k}_x) = \frac{(2\bar{k}_x^2 - n_T^2)^2 + 4\bar{k}_x^2 \bar{k}_{zSL}\bar{k}_{zST} - (\rho/\rho_s) n_T^4 (\bar{k}_{zSL}/\bar{k}_z)}{(2\bar{k}_x^2 - n_T^2)^2 + 4\bar{k}_x^2 \bar{k}_{zSL}\bar{k}_{zST} + (\rho/\rho_s) n_T^4 (\bar{k}_{zSL}/\bar{k}_z)}, \tag{10}$$

where n_L and n_T denote the refraction indices of longitudinal and shear waves, respectively.

Finally, the wave numbers in the z direction, linked to the fluid and the solid are

$$\begin{aligned} \bar{k}_z &= \sqrt{1 - \bar{k}_x^2}, \\ \bar{k}_{zSL} &= \sqrt{n_L^2 - \bar{k}_x^2}, \\ \bar{k}_{zST} &= \sqrt{n_T^2 - \bar{k}_x^2}. \end{aligned} \tag{11}$$

They depend on the domain of definition of $\bar{k}x_i$, and are chosen as real positive when $|\bar{k}_x| < 1, n_L, n_T$ and imaginary positive when $|\bar{k}_x| > 1, n_L, n_T$ (for the usual case $c < c_{ST} < c_{SL}$).

In this case, the reflection coefficient $R(\bar{k}_x)$ has one pole in the complex $\bar{k}_x$ plane, at $\bar{k}_x = \bar{k}_p$ (there is also another pole at $\bar{k}_x = -\bar{k}_p$). With the assumption $\rho/\rho_s \ll 1$, which is the common situation, $\bar{k}_p$ may be written in the form:

$$\bar{k}_p = \bar{k}_R + i\bar{\alpha}_R, \quad \bar{\alpha}_R \ll 1, \tag{12}$$

where $\bar{k}_R = \sin\theta_R$ (θ_R is the Rayleigh angle) and $\bar{\alpha}_R$ is proportional to ρ/ρ_s. Thus, the pole $\bar{k}_p$ is near the real axis of the complex $\bar{k}_x$ plane.

Following a Laurent series expansion about the Rayleigh pole, the reflection coefficient may be written in the approximated form:

$$R(\bar{k}_x) = \frac{\bar{k}_x - \bar{k}_0}{\bar{k}_x - \bar{k}_p}. \tag{13}$$

In the absence of losses in the medium, the zero of the reflection coefficient $\bar{k}_0$ is the complex conjugate $\bar{k}_p^*$ of the pole $\bar{k}_p$.

In order to evaluate analytically the integral (3) we need to express the reflected field in the (x',z') coordinate system where Eq. (9) is written in a more convenient form:

$$P_{\text{ref}}(\bar{x}',\bar{z}') = \frac{\rho c V_0 \sqrt{ka}}{2\sqrt{i\pi}\sin\theta_0} \int_{-\infty}^{+\infty} R(\bar{k}_{x'}) \times \frac{\exp[-(\bar{k}_{x'}^2/4\sin^2\theta_0)]}{\bar{k}_{z'}} e^{i(ka)f_r(\bar{k}_{x'})}\, d\bar{k}_{x'}, \tag{14}$$

where

$$f_r(\bar{k}_{x'}) = \frac{\bar{k}_{x'}^2}{4\sin\theta_0} + \bar{A}_i\bar{k}_{x'} + \bar{B}_i\bar{k}_{z'} \tag{15}$$

with

$$\begin{aligned} \bar{A}_i &= \bar{x}'\cos 2\theta_I + \bar{z}'\sin 2\theta_I, \\ \bar{B}_i &= \bar{x}'\sin 2\theta_I - \bar{z}'\cos 2\theta_I + \bar{d}/\cos\theta_I. \end{aligned} \tag{16}$$

In the neighborhood of the pole, the reflection coefficient $R(\bar{k}_{x'})$ is given by the approximated expression:

$$R(\bar{k}_{x'}) = \frac{\bar{k}_{x'} - \bar{k}_{p'}^*}{\bar{k}_{x'} - \bar{k}_{p'}}, \tag{17}$$

where $\bar{k}_{p'}^*$ is the complex conjugate of the pole $\bar{k}_{p'}$.

In the (x',z') coordinate system the Rayleigh pole is given by

$$\bar{k}_{p'} = \sin(\theta_I - \theta_R) + i(\bar{\alpha}_R/\cos\theta_R)\cos(\theta_I - \theta_R). \tag{18}$$

We note that $\bar{k}_{R'} = \sin(\theta_I - \theta_R)$ is the real part of this pole.

For the special case of incidence at the Rayleigh angle, the pole becomes purely imaginary:

$$\bar{k}_{p'} = i(\bar{\alpha}_R/\cos\theta_R). \tag{19}$$

It is always possible to develop the reflection coefficient in phase and modulus:

$$R(\bar{k}_{x'}) = \rho(\bar{k}_{x'})e^{i\varphi(\bar{k}_{x'})}. \tag{20}$$

Figure 4(a) and (b) shows the modulus, $\rho(\bar{k}_x)$, and the phase, $\varphi(\bar{k}_x)$, of the reflection coefficient, respectively, in the case of a water–aluminum interface. In the neighborhood of the Rayleigh pole (which is a singularity in the complex plane), the modulus of the reflection coefficient tends towards infinity (when $\bar{k}_{x'} \to \bar{k}_{p'}$ we have $|R| \to +\infty$), whereas the phase is regular. In the neighborhood of the Rayleigh angle of incidence the modulus of the reflection coefficient remains regular, but the phase varies rapidly. In order to apply an asymptotic method it is thus necessary to regroup the phase with the function f_r (Ref. 10):

$$\hat{f}_r(\bar{k}_{x'}) = f_r(\bar{k}_{x'}) + \frac{\varphi(\bar{k}_{x'})}{ka}. \tag{21}$$

The function $\hat{f}_r$, being an argument function, is not holomorphic. Thus we cannot apply the steepest descent method since this asymptotic technique, based on the Cauchy theorem, is only applicable to holomorphic functions. However, we may assume $\bar{k}_{x'}$ real, and by making a restriction on the real axis, we may evaluate the integral (14) through application of the asymptotic method of stationary phase.

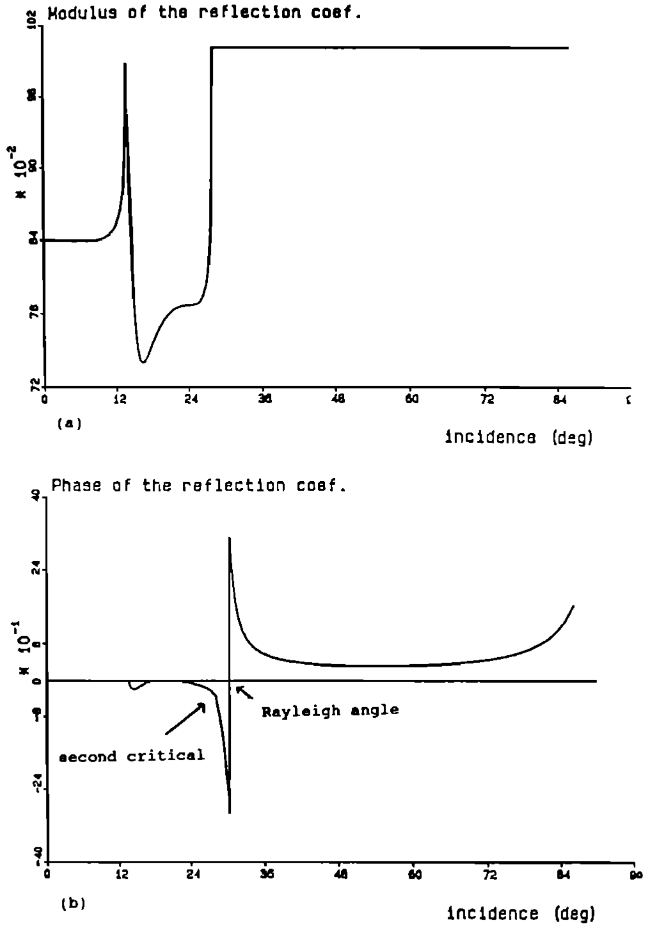


FIG. 4. (a) Modulus of the reflection coefficient. (b) Phase of the reflection coefficient (fluid–solid plane interface).

II. APPLICATION OF THE METHOD OF STATIONARY PHASE

A. Analytical expression of the reflected pressure field

The principle of stationary phase asserts that, as $ka \to +\infty$, the dominant terms in the asymptotic expansion of the integral (14) where $\hat{f}_r(\bar{k}_{x'})$ is real, arise from the immediate neighborhood of the points at which the phase $(ka)\hat{f}_r(\bar{k}_{x'})$ is stationary. We suppose that the coefficients that intervene in the function $\hat{f}_r$ (i.e., $\bar{A}_i,\bar{B}_i$) are of the order 1. Hence, the application of the method consists of calculating the stationary points of the function $\hat{f}_r$, given by $\hat{f}_r'(\bar{k}_{x'}) = 0$, for $\bar{k}_{x'} \in]-1,+1[$.

Let us consider a point of the specular reflected beam of coordinates $(\bar{x}_0',\bar{z}_0')$; the reflected pressure corresponding to this point is given by the integral (14). To the chosen point corresponds a set $\hat{\gamma}_{i0}$ of stationary points calculated by the equation $\hat{f}_r'(\hat{\gamma}_{i0}) = 0$, $i = 1$ or 3.

1. Case of a single stationary point $\hat{\gamma}_{10}$

Here, the integral (14) may be written as:

$$P_{\text{ref}}(\bar{x}_0',\bar{z}_0') = \frac{\rho c V_0\sqrt{ka}}{2\sqrt{i\pi}\sin\theta_0} \int_{\hat{\gamma}_{10}-\epsilon}^{\hat{\gamma}_{10}+\epsilon} \rho(\bar{k}_{x'}) \frac{\exp(-\bar{k}_{x'}^2/4\sin^2\theta_0)}{\bar{k}_{z'}}$$

$$\times e^{i(ka)\hat{f}_r(\bar{k}_{x'})}\, d\bar{k}_{x'}. \tag{22}$$

If the stationary point $\hat{\gamma}_{10}$ is situated near $\bar{k}_{R'}$, the function $\varphi'(\bar{k}_{x'})$, in the interval $]\,\hat{\gamma}_{10}-\epsilon, \hat{\gamma}_{10}+\epsilon\,[$, must be taken into account. At this point, the acoustic field is modified relative to its geometric value:

$$P_{\rm ref}(\bar{x}_0',\bar{z}_0') = \frac{\rho c V_0 e^{-i\pi/2}\rho(\hat{\gamma}_{10})}{\sqrt{-2\sin\theta_0 \hat{f}_r''(\hat{\gamma}_{10})}} \times \frac{\exp[-\hat{\gamma}_{10}^2/4\sin^2\theta_0 + i(ka)\hat{\gamma}_{10}^2/4\sin\theta_0]}{\sqrt{1-\hat{\gamma}_{10}^2}} \times \exp\left[i(ka)(\hat{\gamma}_{10}\bar{A}_{r0} + \sqrt{1-\hat{\gamma}_{10}^2}\,\bar{B}_{r0})\right] + O(1/ka). \tag{23}$$

If the stationary point $\hat{\gamma}_{r0}$ is situated far from $\bar{k}_{R'}$, the function $\varphi'(\bar{k}_{x'})$, in the interval $]\,\hat{\gamma}_{10}-\epsilon, \hat{\gamma}_{10}+\epsilon\,[$, has a negligible value. Thus $\hat{f}_r = f_r$ and $\gamma_{10}=\gamma_{10}$. At this point, the acoustic pressure is specular.

2. Case of three stationary points $\hat{\gamma}_{r0}$

Here, the integral (14) is written:

$$P_{\rm ref}(\bar{x}_0',\bar{z}_0') = \frac{\rho c V_0\sqrt{ka}}{2\sqrt{i\pi}\sin\theta_0} \times\left(\int_{\hat{\gamma}_{10}-\epsilon}^{\hat{\gamma}_{10}+\epsilon} J(\bar{k}_{x'})d\bar{k}_{x'} + \int_{\hat{\gamma}_{20}-\epsilon}^{\hat{\gamma}_{20}+\epsilon} J(\bar{k}_{x'})d\bar{k}_{x'} + \int_{\hat{\gamma}_{30}-\epsilon}^{\hat{\gamma}_{30}+\epsilon} J(\bar{k}_{x'})d\bar{k}_{x'}\right)$$

with

$$J(\bar{k}_{x'}) = \rho(\bar{k}_x)\,\frac{\exp(-\bar{k}_{x'}^2/4\sin^2\theta_0)}{\bar{k}_{z'}}\, e^{i(ka)\hat{f}_r(\bar{k}_{x'})}. \tag{24}$$

If all the stationary points $\hat{\gamma}_{r0}$ satisfy $\varphi'(\hat{\gamma}_{r0}) \ll ka$, the function $\varphi'(\bar{k}_{x'})$ is negligible in the neighborhood of each stationary point; thus, γ_{r0} (which are roots of f_r') also satisfy the equation $\hat{f}_r'=0$. In this case, the reflected pressure corresponding to the point $(\bar{x}_0',\bar{z}_0')$ is specular.

If all the stationary points $\hat{\gamma}_{r0}$ satisfy $\varphi'(\hat{\gamma}_{r0}) \sim ka$, the function $\varphi'(\bar{k}_{x'})$ must be taken into account.

Thus, the analytical expression of the integral (14) is

$$\begin{aligned} P_{\rm ref}(\bar{x}_0',\bar{z}_0') = {} & \frac{\rho c V_0\rho(\hat{\gamma}_1)}{\sqrt{-2\sin\theta_0\hat{f}_r''(\hat{\gamma}_1)}}\, \frac{\exp\left(\dfrac{-\hat{\gamma}_1^2}{4\sin^2\theta_0} + \dfrac{i(ka)\hat{\gamma}_1^2}{4\sin\theta_0} - \dfrac{i\pi}{2}\right)}{\sqrt{1-\hat{\gamma}_1^2}} \exp\left[i(ka)(\hat{\gamma}_1\bar{A}_{r0} + \sqrt{1-\hat{\gamma}_1^2}\,\bar{B}_{r0})\right] \\ & + \frac{\rho c V_0\rho(\hat{\gamma}_2)}{\sqrt{2\sin\theta_0\hat{f}_r''(\hat{\gamma}_2)}}\, \frac{\exp\left(\dfrac{-\hat{\gamma}_2^2}{4\sin^2\theta_0} + \dfrac{i(ka)\hat{\gamma}_2^2}{4\sin\theta_0}\right)}{\sqrt{1-\hat{\gamma}_2^2}} \exp\left[i(ka)(\hat{\gamma}_2\bar{A}_{r0} + \sqrt{1-\hat{\gamma}_2^2}\,\bar{B}_{r0})\right] \\ & + \frac{\rho c V_0\rho(\hat{\gamma}_3)}{\sqrt{2\sin\theta_0\hat{f}_r''(\hat{\gamma}_3)}}\, \exp\frac{\left(-\dfrac{\hat{\gamma}_3^2}{4\sin^2\theta_0} + \dfrac{i(ka)\hat{\gamma}_3^2}{2\sin\theta_0} - \dfrac{i\pi}{2}\right)}{\sqrt{1-\hat{\gamma}_3^2}} \exp\left[i(ka)(\hat{\gamma}_3\bar{A}_{r0} + \sqrt{1-\hat{\gamma}_3^2}\,\bar{B}_{r0})\right] + O\left(\frac{1}{ka}\right). \end{aligned} \tag{25}$$

At this point, the corresponding acoustic field would be modified.

We give some definitions.

A point in the physical space is called "pure" if all the three stationary points involved are roots of the same function $\hat{f}_r'$. This is a point with singularity (in the sense of proximity between the pole and the complex root of $\hat{f}_r$) of order 3; in this case, there are three rays at the Rayleigh angle of incidence.

A point is called "ordinary" if all the three stationary points involved are roots of the same function f_r'.

Let us consider the case of a point having three stationary points $\hat{\gamma}_{r0}$, of which only one or two are in the neighborhood of $\bar{k}_{R'}$, and the other(s) are situated outside this area. For this point, the acoustic pressure field will be modified relative to its geometric value. This case corresponds to a physical point inside the focal zone of the nonspecular reflected field. This point is defined by only two (or one) specular rays, and one (or two) nonspecular ray(s). This is a point of singularity of order 2 (or order 1). We call this point "impure" as opposed to the definition of the "pure" point.

In the case of an impure point, the corresponding acoustic pressure will be modified relative to its geometric value.

Remark: The expressions (23) and (25) give the pressure field at any nonsingular point in the physical space. The solution diverges on the caustic and at the focal point, and we have to apply the method of Ludwig,[11] as in Refs. 9 and 10. The technique used together with the expressions of the pressure field are not given here as they are analogous.

In order to evaluate the expressions (23) and (25) we have to calculate the stationary points $\hat{\gamma}_{i0}$ as a function of γ_{i0}; this is discussed in the following section.

B. Study of stationary points

In the case of the reflection of a focused beam at a plane interface in the neighborhood of the Rayleigh angle of incidence, the generation of a Rayleigh wave causes a small displacement of the stationary points of the specular beam (i.e., a small displacement of the geometrical reflected rays).

In a given point, the new stationary points are roots of the function $\hat{f}'_r(\bar{k}_{x'}) = f'_r(\bar{k}_{x'}) + \varphi'(\bar{k}_{x'})/ka$, where it is supposed that

$$1 \sim \varphi'(\bar{k}_{x'})/ka \ll f'_r(\bar{k}_{x'}). \tag{26}$$

For reasons reported at the end of Sec. II A, the following analysis is applied for points that do not change their "physical nature." A point initially situated in the interior (conversely in the exterior) of the specular focal zone, changes its physical nature if it is found in the exterior (conversely in the interior) of the nonspecular focal zone; to the considered point initially correspond three stationary points (conversely one single real root) and finally corresponds one single stationary point (conversely three real roots). Points in the neighborhood of the caustic are liable to change their physical nature.

Let us consider two points, M_1 and M_2, in the exterior and in the neighborhood of the specular reflected caustic [see Fig. 5(a)]. For these points, the incidence of the beam is close to the Rayleigh angle.

The rays with an incidence near the Rayleigh angle will be slightly modified.

The point M_1 corresponds to a ray with an incidence far from the Rayleigh angle. This point will not change its physical nature. Let us see the position of the roots of $\hat{f}'_r$ in the complex plane [see Fig. 5(b)]. The rectangle indicates the influence area of φ'.

The two complex roots situated far from this area, will not be displaced and thus will not become real. The point M_2 corresponds to a ray with an incidence near the Rayleigh angle. This point can change its physical nature. The position of the roots in the complex plane is shown in Fig. 5(c).

In this case, the two complex roots situated near the area of the Rayleigh influence, can be slightly displaced and become real. This will result in a distortion of the caustic of the reflected beam. The caustic will be displaced to the right (see Fig. 9). A curvature in the acoustic axis is thus observed.

In region I (where the point M_1 is situated) the gradient of the acoustic pressure would be lower than its geometrical value. In region II (where the point M_2 is situated) the gradient of the acoustic pressure would be greater than its geometrical value. Thus we observe an asymmetric variation of the reflected pressure field around the axis.

For points far from the specular caustic, the stationary points $\hat{\gamma}_{i0}$ are in the neighborhood of the stationary points γ_{i0}. Hence, we can calculate $\hat{\gamma}_{i0}$ from a Taylor expansion of functions $f'_r(\hat{\gamma}_{i0})$ and $\varphi'(\hat{\gamma}_{i0})$ about γ_{i0}:

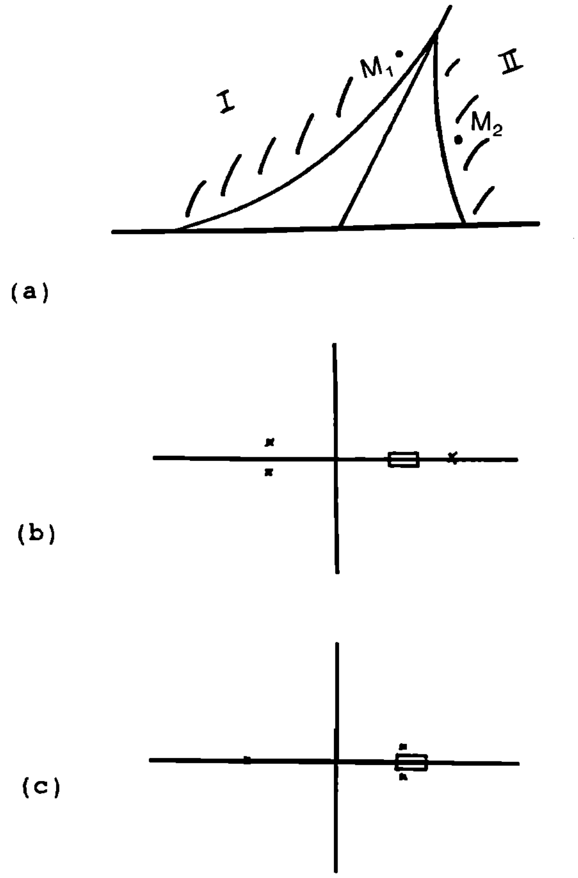


FIG. 5. (a) Points M_1 and M_2 in the exterior of the specular reflected beam. (b) Point M_1: location of the three roots of f'_r in the complex plane. (c) Point M_2: location of the three roots of f'_r in the complex plane.

$$f'_r(\hat{\gamma}_{i0}) = f'_r(\gamma_{i0}) + f''_r(\gamma_{i0})(\hat{\gamma}_{i0} - \gamma_{i0}),$$
$$\varphi'(\hat{\gamma}_{i0}) = \varphi'(\gamma_{i0}) + \varphi''(\gamma_{i0})(\hat{\gamma}_{i0} - \gamma_{i0}).$$

Then

$$\begin{aligned}\hat{f}'_r(\hat{\gamma}_{i0}) &= f'_r(\hat{\gamma}_{i0}) + \varphi'(\hat{\gamma}_{i0})/ka \\ &= f'_r(\gamma_{i0}) + f''_r(\gamma_{i0})(\hat{\gamma}_{i0} - \gamma_{i0}) \\ &\quad + \frac{\varphi'(\gamma_{i0})}{ka} + \frac{\varphi''(\gamma_{i0})}{ka}(\hat{\gamma}_{i0} - \gamma_{i0}) = 0\end{aligned}$$

and thus

$$\hat{\gamma}_{i0} = \gamma_{i0} - \frac{\varphi'(\gamma_{i0})/ka}{f''_r(\gamma_{i0}) + \varphi''(\gamma_{i0})/ka} = \gamma_{i0} - \epsilon',$$

where ϵ' is a small parameter [hypothesis (26)].

C. Deformation of the reflected field

In the preceding section we deduced an asymmetric variation of the reflected pressure around the caustic of the beam (displacement of the reflected caustic to the right—see Fig. 9). Our aim now is to reconstruct, point by point, the new caustic taking into account the nonspecular phenomena, and thus determine the position of the new focal point.

Let us consider a point $(\bar{A}_{i0}, \bar{B}_{i0})$ of the reflected geometric beam. This point corresponds to a set of stationary points γ_{i0}. If the chosen point $(\overline{A}_{i0},\overline{B}_{i0})$ is situated in the focal zone, we have the case of three stationary points (γ_{10}, γ_{20}, γ_{30}), which satisfy the equations:

$$f'_r(\gamma_{10})=0:\quad \frac{\gamma_{10}}{2\sin\theta_0}+\overline{A}_{i0}-\frac{\overline{B}_{i0}\gamma_{10}}{\sqrt{1-\gamma_{10}^2}}=0, \tag{27a}$$

$$f'_r(\gamma_{20})=0:\quad \frac{\gamma_{20}}{2\sin\theta_0}+\overline{A}_{i0}-\frac{\overline{B}_{i0}\gamma_{20}}{\sqrt{1-\gamma_{20}^2}}=0, \tag{27b}$$

$$f'_r(\gamma_{30})=0:\quad \frac{\gamma_{30}}{2\sin\theta_0}+\overline{A}_{i0}-\frac{\overline{B}_{i0}\gamma_{30}}{\sqrt{1-\gamma_{30}^2}}=0. \tag{27c}$$

We calculate the coordinates of a new point $(\overline{\hat{A}}_{i0},\overline{\hat{B}}_{i0})$ to which correspond the same stationary points γ_{10}, γ_{20}, γ_{30} (we suppose $\hat{\gamma}_{10}=\gamma_{10}$, $\hat{\gamma}_{20}=\gamma_{20}$, $\hat{\gamma}_{30}=\gamma_{30}$). We call this point the "image" of the point $(\overline{A}_{i0},\overline{B}_{i0})$ in the nonspecular beam.

By expanding $\varphi'(\overline{k}_{x'})$ about the stationary points γ_{i0}, situated in the neighborhood of $\overline{k}_{R'}$ and supposed equivalents, the three stationary points satisfy the equations:

$$\hat{f}'_r(\gamma_{10})=0:\quad \left(\frac{1}{2\sin\theta_0}+N\right)\gamma_{10}+\overline{\hat{A}}_{i0}+M-\frac{\overline{\hat{B}}_{i0}\gamma_{10}}{\sqrt{1-\gamma_{10}^2}}=0, \tag{28a}$$

$$\hat{f}'_r(\gamma_{20})=0:\quad \left(\frac{1}{2\sin\theta_0}+N\right)\gamma_{20}+\overline{\hat{A}}_{i0}+M-\frac{\overline{\hat{B}}_{i0}\gamma_{20}}{\sqrt{1-\gamma_{20}^2}}=0, \tag{28b}$$

$$\hat{f}'_r(\gamma_{30})=0:\quad \left(\frac{1}{2\sin\theta_0}+N\right)\gamma_{30}+\overline{\hat{A}}_{i0}+M-\frac{\overline{\hat{B}}_{i0}\gamma_{30}}{\sqrt{1-\gamma_{30}^2}}=0, \tag{28c}$$

where

$$M=\frac{1}{n}\sum_{i=1}^{n}\frac{\varphi'(\gamma_{i0})-\gamma_{i0}\varphi''(\gamma_{i0})}{ka}$$

and

$$N=\frac{1}{n}\sum_{i=1}^{n}\frac{\varphi''(\gamma_{i0})}{ka},$$

n denoted the number of stationary points near $\overline{k}_{R'}$; the functions φ' and φ'' are, respectively, the first and second derivatives of the phase φ of the reflection coefficient. By subtracting Eqs. (27a) − (28a), (27b) − (28b), (27c) − (28c) we obtain a system of three equations that can be used to determine $\overline{\hat{A}}_{i0}$ and $\overline{\hat{B}}_{i0}$:

$$N\gamma_{10}+(\overline{\hat{A}}_{i0}-\overline{A}_{i0})+M=(\overline{\hat{B}}_{i0}-\overline{B}_{i0})\frac{\gamma_{10}}{\sqrt{1-\gamma_{10}^2}}, \tag{29a}$$

$$N\gamma_{20}+(\overline{\hat{A}}_{i0}-\overline{A}_{i0})+M=(\overline{\hat{B}}_{i0}-\overline{B}_{i0})\frac{\gamma_{20}}{\sqrt{1-\gamma_{20}^2}}, \tag{29b}$$

$$N\gamma_{30}+(\overline{\hat{A}}_{i0}-\overline{A}_{i0})+M=(\overline{\hat{B}}_{i0}-\overline{B}_{i0})\frac{\gamma_{30}}{\sqrt{1-\gamma_{30}^2}}. \tag{29c}$$

From Eqs. (29a) and (29b) we obtain:

$$\overline{\hat{B}}_{i0}=\overline{B}_{i0}+\frac{N(\gamma_{10}-\gamma_{20})}{\gamma_{10}/\sqrt{1-\gamma_{10}^2}-\gamma_{20}/\sqrt{1-\gamma_{20}^2}}, \tag{30}$$

$$\overline{A}_{io}=\overline{A}_{i0}-M-N\gamma_{10}+N(\gamma_{10}-\gamma_{20})\gamma_{10}\times\left[\frac{\gamma_{10}}{\sqrt{1-\gamma_{10}^2}}\left(\frac{\gamma_{10}}{\sqrt{1-\gamma_{10}^2}}-\frac{\gamma_{20}}{\sqrt{1-\gamma_{20}^2}}\right)\right]^{-1}. \tag{31}$$

Remark: The determinant of these three equations is zero, and the third equation is a linear combination of other two.

Equations (30) and (31) give the nonspecular image of a point of the geometrical beam, if the initial point is not situated on the specular caustic (i.e., the initial point must correspond to three distinct stationary points).

If the initial point is a point of the specular caustic, which corresponds to the set of stationary points: $\gamma_{10}=\gamma_{20}$, γ_{30}, this set satisfies the equations $f'_r=f''_r=0$ and thus we obtain

$$\overline{A}_{i0}^{2/3}+\overline{B}_{i0}^{2/3}=(1/2\sin\theta_0)^{2/3}. \tag{32}$$

We calculate the coordinates of an image point of the nonspecular reflected caustic that corresponds to the same set of stationary points $\gamma_{10}=\gamma_{20}$, γ_{30}; these stationary points also satisfy the equations $\hat{f}'_r=\hat{f}''_r=0$.

Thus

$$\left[\overline{\hat{A}}_{i0}+\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\varphi'(\gamma_{i0})-\gamma_{i0}\varphi''(\gamma_{i0})}{ka}\right)\right]^{2/3}+\overline{\hat{B}}_{i0}^{2/3}=\left(\frac{1}{2\sin\theta_0}+\frac{1}{n}\sum_{i=1}^{n}\frac{\varphi''(\gamma_{i0})}{ka}\right)^{2/3}. \tag{33}$$

The coordinates of the image point are

$$\overline{\hat{A}}_{i0}=\overline{A}_{i0}-M-N\gamma_{i0}^3,\qquad \overline{\hat{B}}_{i0}=\overline{B}_{i0}+N(1-\gamma_{i0}^2)^{3/2}. \tag{34}$$

From the system $f'_r=f''_r=f'''_r=0$, we deduce three stationary points equal to zero corresponding to the specular focal point. Thus, the coordinates of the geometrical focal point are

$$\overline{A}_{i0}=0,\quad \overline{B}_{i0}=1/2\sin\theta_0. \tag{35}$$

From the system $\hat{f}'_r=\hat{f}''_r=\hat{f}'''_r=0$, we also denote three stationary points equal to zero, corresponding to the new focal point. Thus, the coordinates of the new (nonspecular) focal point are

$$\overline{\hat{A}}_{i0}=-\frac{\varphi'(0)}{ka},\quad \overline{\hat{B}}_{i0}=\frac{1}{2\sin\theta_0}+\frac{\varphi''(0)}{ka}. \tag{36}$$

By comparing the expressions (35) and (36), we deduce the lateral L and axial A displacements of the focal point:

$$L=-\frac{\varphi'(0)}{ka}=-\frac{\tilde{\varphi}'(\overline{k}_t)\cos\theta_t}{ka}, \tag{37}$$

$$A = -\frac{\varphi''(0)}{ka} = \frac{\tilde{\varphi}''(\bar{k}_I)\cos^2\theta_I - \tilde{\varphi}'(\bar{k}_I)\sin\theta_I}{ka}, \tag{38}$$

where $\varphi'(0)$ and $\varphi''(0)$ are the first and second derivatives of the phase of the reflection coefficient, in the system (x',z'); where $\varphi'(\bar{k}_I)$ and $\varphi''(\bar{k}_I)$ are the first and second derivatives of the phase of the reflection coefficient calculated at the point $\bar{k}_I = \sin\theta_I$, in the (x,z) coordinate system.

The first derivative is negative, thus the lateral displacement always remains positive. For the Rayleigh angle incidence, for which $\bar{k}_I = \bar{k}_R$, L becomes maximum. There is no axial displacement for an angle of incidence noted θ_{I0}, which is defined by the equation:

$$\tilde{\varphi}''(\sin\theta_{I0})\cos^2\theta_{I0} - \tilde{\varphi}'(\sin\theta_{I0})\sin\theta_{I0} = 0. \tag{39}$$

In practice, the value of θ_{I0} is near θ_R.

For incidence $\theta_I < \theta_{I0}$, A is negative with a minimum for an incidence denoted $\theta_{\min}$; for incidence $\theta_I > \theta_{I0}$, A is positive with a maximum for an incidence denoted $\theta_{\max}$. The results for this particular case of our study conform to those reported by Bertoni *et al.*[5]

In Fig. 6(a) and (b), L and A are represented qualitatively as a function of the incidence. In Fig. 7(a) and (b), we represent L and A quantitatively for an emitter at 5 MHz. These theoretical results conform to our experimental data that is presented in a companion paper.[12]

Remark: For an ordinary point, image and initial points are superimposed. We can find both the image of points of the caustic and of points in the interior of this caustic. The part of the caustic situated far from the focal point (ordinary points) will not be distorted; for this region the image corresponds to the initial point.

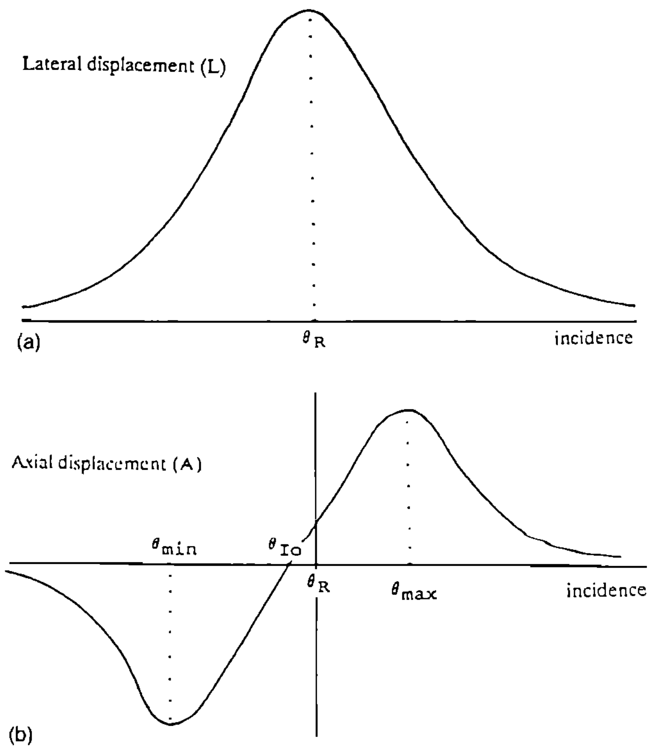


FIG. 6. (a) Qualitative lateral displacement of the focal point. (b) Qualitative axial displacement of the focal point.

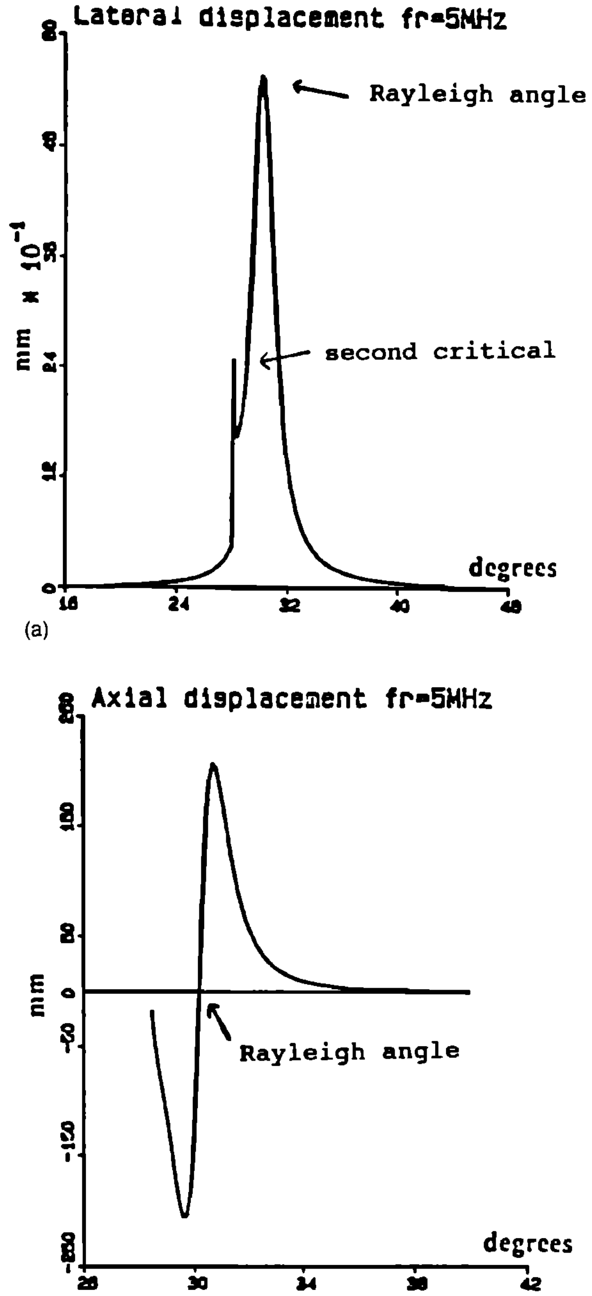
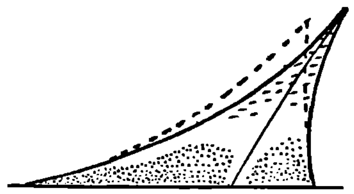


FIG. 7. (a) Lateral displacement of the focal point (emitter at 5 MHz). (b) Axial displacement of the focal point (emitter at 5 MHz).

Figure 8 shows the regions of pure and impure points in the focal zone of the nonspecular reflected beam. In the case of an initial point situated in the exterior of the focal zone (this point corresponds to a single stationary point), there is only one equation to calculate the coordinates of the image. Thus, we find a direction on which the image is situated.

During our analysis, we have already seen that in the neighborhood of $\bar{k}_R$, the phase of the reflection coefficient, $\varphi(\bar{k}_{x'})$, varies abruptly and thus $\varphi'(\bar{k}_{x'})$ must be taken into account. In the exterior of this neighborhood, the phase re-



/ / / zone of saddle-points $\hat{\gamma}_{i0}$ (pure points)

zone of impure points (mixed saddle-points $\hat{\gamma}_{i0}$ and γ_{i0})

∴∴ zone of saddle-points γ_{i0} (ordinary points)

FIG. 8. Nonspecular reflected beam: regions of pure, impure, and ordinary points.

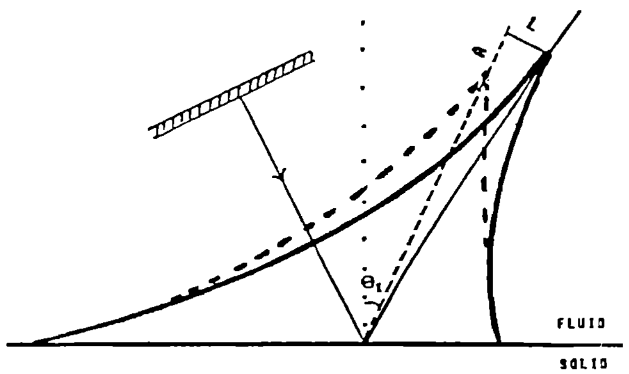


FIG. 9. Qualitative representation of the distortion of the caustic of the reflected beam: - - - - geometrical caustic; ——— distorted caustic.

mains constant and thus $\varphi'(\bar{k}_{x'})$ is negligible.

Now we will consider two cases.

(a) Case of an incidence near the Rayleigh angle: $|\theta_I - \theta_R| \sim \epsilon$. Here, $\bar{k}_{R'} = \sin(\theta_I - \theta_R)$ tends to zero, and is zero for incidence at the Rayleigh angle.

We distinguish two possibilities.

(*a1*) If at least one stationary point y_i remains in the neighborhood of $\bar{k}_{R'}$, among the points of the physical space concerned here are the following.

(i) A region around and on the focal point (for which the three stationary points are equal to zero): lateral and axial displacements.

(ii) A region around and on the part of the caustic situated near the focal point (where the three stationary points are near $\bar{k}_R$, which is near zero): distortion of the caustic, spreading of the reflected beam, asymmetric variation of the acoustic pressure around the reflected caustic.

(iii) A region around and on the entire acoustic axis (for which one stationary point is always equal to zero): curvature of the acoustic axis. In Fig. 9, the regions of the geometrical caustic (a broken line) for the case (a1) are illustrated. For incidence near the Rayleigh angle, these regions of the reflected beam would be modified (nonspecular reflection).

(*a2*) If we consider a set of stationary points situated far from $\bar{k}_{R'}$, $|\gamma_i - \bar{k}_{R'}| \gg \epsilon$, in spite of an incidence in the neighborhood of the Rayleigh angle, this set of stationary points corresponds to a point of the specular beam.

(b) In the case of an incidence far from the Rayleigh angle, $|\theta_I - \theta_R| \gg \epsilon$, we can distinguish two possibilities.

(*b1*) If the stationary points are situated far from $\bar{k}_{R'}$, $|\gamma_i - \bar{k}_{R'}| \gg \epsilon$, they correspond to points of the specular reflected beam.

(*b2*) If at least one stationary point γ_i remains in the neighborhood of $\bar{k}_{R'}$, $|\gamma_i - \bar{k}_{R'}| \ll \epsilon$, in spite of an incidence far from the Rayleigh angle, there are points in the physical space for which the reflected beam would be modified with respect to values defined by geometrical acoustics.

A part of the caustic situated far from the focal point will be modified. We note that the focal point (three stationary points equal to zero) is not of concern here because of the general assumption (b) of an incidence far from the Rayleigh angle (i.e., $\bar{k}_{R'}$ is far from zero).

III. CONCLUSIONS

In this paper we have studied the structure of the acoustic field in the case of the reflection of focused Gaussian beams from a fluid–solid interface. Initially, the incident beam was modeled by plane wave decomposition using the Fourier integral representation. Then, the reflected beam was described in a coordinate system corresponding to the emitter. In this system, the expression of the phase function is simplified, so it was possible to calculate its stationary points analytically. The reflected pressure field was finally deduced by means of the asymptotic method of stationary phase, based on the short wave hypothesis.

The analytical expression obtained is valid for any angle of incidence. A modification of the structure of the reflected focused beam, in relation to the provisions of the geometrical acoustics, was observed for any incidence. In the specific case of an incidence in the neighborhood or equal to the Rayleigh angle, nonspecular phenomena involve a part of the caustic of the reflected beam including the focal point, and the acoustic axis in its entirety. Simple expressions of the axial and lateral displacements of the focal point were deduced, and led to a numeric quantification. The theoretical results obtained here are in agreement with our experimental data reported in a companion paper.[12]

In the case of a focused reflected beam, only a part of the energy falls on the interface at the Rayleigh angle. The abrupt variation of the phase of the reflection coefficient in the neighborhood of the Rayleigh angle, causes a local modification of the reflected beam. Further, for a focused beam, rays always fall on the interface at the Rayleigh angle even if the beam incidence is far from that angle. For this reason, there are always regions of the reflected field that are modified (even if the beam incidence is far from the Rayleigh angle). However, in the case of a bounded reflected beam,[13] and in the context of geometric acoustics, all energy (except the small quantity of diffracted energy) falls on the interface with the same angle. If the incidence of the beam is equal to the Rayleigh angle, generation of a Rayleigh wave occurs following its reradiation in the fluid resulting the well-known nonspecular phenomena.

ACKNOWLEDGMENTS

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